

Local constitutive parameters of metamaterials

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The reason of the non-locality of constitutive (material) parameters extracted in a usual way from the reflection-transmission coefficients of composite slab at moderately low frequencies is explained. The physical meaning of these parameters is clarified. Local constitutive parameters of metamaterial lattices are discussed and their existence at moderate frequencies is demonstrated. It is shown how to extract local material parameters from the dispersion characteristics of an infinite lattice and from reflection and transmission coefficients of metamaterial layers.

PACS numbers: 78.20.Ci, 42.70.Qs, 42.25.Gy, 73.20.Mf, 78.67.Bf

I. INTRODUCTION

In the modern scientific literature the so-called *metamaterials*^{1,2} correspond to a lot of exciting results and a lot of mess. Metamaterials (MTM) are often defined as media which possess constitutive (material) parameters not observed in nature. Thanks to these material properties, many exotic phenomena were predicted and discovered in the MTM structures (a detailed overview is presented in¹). As a rule, MTM are presented by lattices of reciprocal optically small resonant particles, such as complex-shape metal inclusions (in different frequency ranges), plasmonic and polaritonic nano-spheres and nano-wires (at infrared waves and in the visible). Particles characteristic size δ and maximal period of lattices a at the frequency of particle resonance are usually assumed to be much smaller than the wavelength λ in the lattice matrix. This assumption is usually considered as allowing one to homogenize MTM and to explain their exotic properties through material parameters (MP). However, it is important that δ and a in MTM though small are not negligible with respect to λ . Practically, they lie within the frequency band

$$0.01 < \frac{(a, \delta)}{\lambda} < 0.1. \quad (1)$$

As a result, the effective wavelength in MTM (it can be expressed though the lattice eigenwave wavenumber q as $\lambda_{\text{eff}} = 2\pi/q$) being shortened at the particle resonance compared to λ can approach to $a/2$. At such frequencies the lattice spatial resonance (often called as *Bragg's resonance*) holds. Then the Bragg (staggered) mode which has complex wavenumber $q = \pi/a + j\text{Im}(q)$ can be excited in the lattice. Within the band of the staggered mode the homogenization of the metamaterial lattice is meaningless. If one formally introduces material parameters for such a regime they cannot satisfy to the physical conditions of locality. Basic physical limitations which are equivalent to the concept of locality are as follows (see e.g. in³):

- passivity (for the temporal dependence $e^{-i\omega t}$ it implies $\text{Im}(\varepsilon) > 0$ and $\text{Im}(\mu) > 0$ simultaneously at

all frequencies, for $e^{j\omega t}$ the sign of both $\text{Im}(\varepsilon)$ and $\text{Im}(\mu)$ should be negative);

- causality (in the region of negligible losses the frequency derivatives of both $\text{Re}(\varepsilon)$ and $\text{Re}(\mu)$ should be positive);
- absence of radiation losses (it was postulated for periodic infinite structures in⁴, then derived in⁵, now this principle is often attributed to work⁶).

Constitutive parameters of MTM are non-local at certain frequencies inside the resonant band of the particle. From it one often deduces that it is impossible to introduce local MP over the whole resonant band. In the present paper we will show that this opinion is wrong.

It is evident that the staggered mode of a lattice does not cover the whole region of moderately low frequencies 1, and there are bands where $a < \lambda_{\text{eff}}/2$. There we should expect the locality of measured or simulated MP if they are measured or simulated properly). However, inspecting well-known works^{7,8,9,11,12,13,14} (and many others) devoted to the extraction of material parameters (MP) from measured or simulated reflection (R) and transmission (T) coefficients of a metamaterial slab one observes another result. At least one of two extracted MP in all these works violates all locality conditions over the whole frequency range 1. It will be shown below that the reason of it is the different physical meaning of these MP than the meaning of local constitutive parameters introduced in the quasistatic theory of lattices (e.g.¹⁵). In¹⁶ it was properly noticed that MP introduced for orthorhombic lattices in^{17,18} are non-local *by definition*. In fact, these MP are exactly the same as MP measured or simulated in^{7,8,9,11,12,13,14}. In the present paper these non-local MP will be complemented by local ones extracted from same R and T for the same frequency range defined by 1.

In⁷ one proposed to extract MP of lattices of thickness d as if these were filled by a continuous medium with parameters ε_{eff} , μ_{eff} . Then R and T for the normal incidence take form:

$$R = \frac{r(1 - e^{-2jq_{\text{eff}}d})}{1 - r^2e^{-2jq_{\text{eff}}d}}, \quad T = \frac{e^{-jq_{\text{eff}}d}(1 - r^2)}{1 - r^2e^{-2jq_{\text{eff}}d}}. \quad (2)$$

Here $q_{\text{eff}} = \omega \sqrt{\varepsilon_0 \mu_0 \varepsilon_{\text{eff}} \mu_{\text{eff}}} \equiv k_0 \sqrt{\varepsilon_{\text{eff}} \mu_{\text{eff}}}$ is the wavenumber of the medium filling the layer. Also in these formulas r is the reflection coefficient from semi-infinite medium expressed through Z (medium wave impedance normalized to that of free space) as

$$r = \frac{Z - 1}{Z + 1}, \quad Z = \sqrt{\frac{\mu_{\text{eff}}}{\varepsilon_{\text{eff}}}}. \quad (3)$$

II. LOCAL AND NON-LOCAL MATERIAL PARAMETERETS

In works^{22,23,24,25} it was shown that the approximation of a uniform slab filled by effective continuous medium is too rough within the frequency range 1. This is so even for non-resonant inclusions studied in^{22,23,24,25}, and, moreover, it is so for resonant ones. However, this observation does not mean that the homogenization of slabs at moderately low frequencies is impossible at all. In these works two slightly different algorithms of homogenization were developed. The original composite slab of thickness d was replaced by a three-layer structure. These effective layers are assumed to be filled by different continuous media. The central layer was described through local constitutive parameters ε_L and μ_L which were calculated through the known polarizabilities of particles. Two *transition* layers of small thickness $d_t \approx \delta$ were described by special anisotropic material parameters ε_t and μ_t (also local). For ε_t and μ_t the closed form relations were derived. These parameters allow one to match normal components of averaged electric displacement $\langle \mathbf{D} \rangle$ and magnetic flux $\langle \mathbf{B} \rangle$ at all boundaries. This turned out to be possible up to $qa \approx 1$ and for *all angles of incidence*²⁵. The last fact means that MP of both central and transition layers at fixed frequency do not depend on the direction of propagation. It is an obvious feature of local MP. Using this model the theoretical reflection coefficient for arbitrary number N of lattice unit cells across the layer (starting from $N = 1$) turned out to be practically equal to that obtained by exact simulations^{24,25}.

Though the use of local MP^{22,23,24,25} makes the problem of the plane-wave reflection in the frequency range 1 more difficult these parameters are, at least, as important as non-local MP. Once extracted, tensors of local MP are independent on the wave incidence angle and polarization. Non-local MP at the same frequency should be measured or simulated separately for all possible directions of the wave vector. Local MP can describe the interaction between the medium with wave packages, moreover, with evanescent waves. Non-local MP are applicable for the only propagating wave. For the only wave they do not violate the passivity condition in the effective medium. In a plane wave electric and magnetic fields are related through the wave impedance. Then the negative electric or magnetic losses (wrong sign of ε_{eff} for magnetic MTM⁷⁻¹⁴ or, vice versa, wrong sign of μ_{eff} for dielectric MTM) are compensated by the positive magnetic or

electric losses. A similar speculation can be done on the causality. For wave packages and for evanescent waves it is not so because in this case the electric field and magnetic field can be locally decoupled. For example, in the near zone of a wire antenna its electric field dominates. Then the wrong sign of electric losses in the substrate implies the energy generation, and the effective medium turns out to be active. It is also possible to show the violation of the causality in these cases. So, non-local MP give not enough insight on MTM, and to know local MP is important.

In the present paper it will be shown how to find local constitutive parameters through R and T of a slab. However, before it, let us understand why MP obtained in⁷⁻¹⁴ and other similar works are non-local. To answer this question one can discuss the meaning of non-local MP in terms of the transmission-line theory. Consider a problem of the plane-wave reflection by a semi-infinite lattice of dipole scatterers. The strict analytical solution of this problem was found in²⁶. For simplicity let us restrict by the special case of normal incidence to a orthorhombic lattice of dipoles located in free space. Also we assume that the frequency range satisfies to the condition $k_0 a = \omega a \sqrt{\varepsilon_0 \mu_0} < \pi$, where a is the lattice period in the direction of propagation (in this frequency region the only eigenwave can propagate). Then the reflection coefficient of the lattice takes form²⁶:

$$r = \frac{\sin \frac{(k_0 - q)a}{2}}{\sin \frac{(k_0 + q)a}{2}} \Pi. \quad (4)$$

Here q is the eigenwave wavenumber and the factor Π for which a closed-form expression was obtained in²⁶ takes into account all polaritons excited at the interface. It was proved in²⁶ that at the frequencies of our interest this factor has unit absolute value. So, it can be represented as an imaginary exponential $\Pi = e^{j\Phi}$. Then the influence of polaritons can be taken in account by a displacement of the interface to which the new reflection coefficient r' is referred with respect to the upper crystal plane to which reflection coefficient r refers in formula 4. After it we can rewrite 4 in the form

$$r' = r \exp(-j\Phi) = \frac{Z_B - 1}{Z_B + 1}, \quad Z_B = \frac{\tan(k_0 a/2)}{\tan(qa/2)}. \quad (5)$$

The phase shift Φ is rather small for $qa < 1$ ²⁶.

If $\max(qa/2, k_0 a/2) \ll 1$ the sine and tangent functions in 4 and 5 can be replaced by their arguments. Then we obtain from 5 $Z_B = k_0/q$ that gives the approximation of the continuous medium. Really, if we compare $Z_B = k_0/q = 1/\sqrt{\varepsilon_{\text{eff}} \mu_{\text{eff}}}$ with 3 we obtain $\mu_{\text{eff}} = 1$, $q = k\sqrt{\varepsilon_{\text{eff}}}$ and 5 becomes the Fresnel formula for the reflection by a continuous dielectric half-space. The non-locality of the effective permittivity extracted through R in this case is negligible. The practical condition of applicability of this approximation follows from numerical properties of tangent function and reads as

$$\max\left(\frac{a}{\lambda}, \frac{a}{\lambda_{\text{eff}}}\right) < 0.01. \quad (6)$$

This condition gives the upper frequency bound up to which formulas 2 and 3 are compatible with approximation of the continuous medium. The frequency region 6 is the same for which transition layers in the theory²⁴ have no visible influence. For practical implementations of MTM the frequencies satisfying 6 are not interesting.

At frequencies 1 the normalized impedance $Z = \sqrt{\varepsilon_{\text{eff}}/\mu_{\text{eff}}}$ can be understood through parameters of a periodically loaded transmission line (TL). Formula 5 obtained as the *strict* solution of the boundary lattice problem is known in the theory of periodically loaded TL. It represents the so-called *Bloch impedance* of a line loaded by shunt lumped impedances with period a (formula (5.117) from²⁷). The Bloch impedance can be defined as the characteristic impedance of the homogenized periodically loaded line¹⁹ normalized to the impedance η of the host matrix. Notice, that the dispersion equation of a TL with shunt loads is known (e.g.²⁸):

$$\cos(qa) = \cos(ka) + \frac{j}{2Z_{\text{load}}} \sin(ka), \quad (7)$$

where k is the matrix wavenumber and Z_{load} is the normalized impedance of loads. The equation 29 coincides with equation (5.226) from²⁷, thoroughly derived for an infinite lattice of electric dipoles at frequencies $ka < \pi$. So, the problem of the reflection and transmission in dipole lattices can be correctly solved replacing the crystal planes by effective lumped loads and using the model of a periodically loaded TL. The reflection coefficient R of a semi-infinite lattice is directly related to its Bloch impedance $Z = Z_B$. If we measure R and T for a composite slab with integer number of cells across it, these parameters will uniquely determine q and Z (and, consequently, non-local MP). Thus, one can conclude that MP calculated through R and T using 2 and 3 are *transmission-line material parameters* (TLMP), introduced for TL networks in 2002 in works^{19,21}. The idea of TLMP can be better understood from the comparison of Figs. 1 and 2.

To understand the reason why TLMP are non-local let us show that Z_B can be expressed in terms of the Bloch expansion for microscopic fields in lattices. Namely, if a linearly polarized eigenmode with wave vector \mathbf{q} propagates along the axis z in a lattice with periods ($b_x, b_y, b_z = a$) the Bloch impedance is equal to the ratio of zero-order spatial harmonics denoted as \mathbf{E}_0 and \mathbf{H}_0 in expansions:

$$\begin{cases} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{cases} = \begin{cases} \mathbf{E}_0 \\ \mathbf{H}_0 \end{cases} e^{-jqz} + \sum_{\mathbf{n} \neq \mathbf{0}} \mathbf{E}_{\mathbf{n}} e^{-j(qz + \mathbf{G}_{\mathbf{n}} \cdot \mathbf{r})}, \quad (8)$$

where $\mathbf{G}_{\mathbf{n}} = (2\pi n_x/b_x, 2\pi n_y/b_y, \frac{2\pi n_z}{a})$ are multiples of generic lattice vector. From 8 we have

$$\begin{cases} \mathbf{E}_0 \\ \mathbf{H}_0 \end{cases} = \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} \begin{cases} \mathbf{E}_{TA}(z) \\ \mathbf{H}_{TA}(z) \end{cases} e^{+jqz} dz, \quad (9)$$

where the index TA means the transverse averaging that allows one to get rid of G_x, G_y in 8. It is evident that these integrals are exactly equal to $E_{TA}(-a/2) = E_{TA}(a/2) \exp(jqa)$, $H_{TA}(-a/2) = H_{TA}(a/2) \exp(jqa)$. The ratio $Z_B = E_{TA}(\pm a/2)/\eta H_{TA}(\pm a/2)$ can be interpreted as that of effective voltage and effective current taken at the input or output of the cell. It is an alternative definition of the normalized Bloch impedance of periodically loaded TL with unit cell length a ¹. The Bloch impedance that determines (together with q) non-local MP of a lattice is equal to: (a) the characteristic impedance of the equivalent homogenized TL, (b) the ratio of fundamental Bloch harmonics of electric and magnetic fields. The last fact is the reason of the non-locality of TLMP.

Really, in the definition of local MP we deal with averaged electric and magnetic fields $\langle \mathbf{E} \rangle$, $\langle \mathbf{H} \rangle$ which are simple integrals of the microscopic field over the domain around the observation point. Integrating Maxwell's equations for microscopic fields does not change these equations and they hold for $\langle \mathbf{E} \rangle$ and $\langle \mathbf{H} \rangle$ and averaged polarizations. However, they do not hold for zeroth Bloch harmonics of microscopic fields taken separately. All high-order terms of expansion 9 give a nonzero contribution into $\langle \mathbf{E} \rangle$ and $\langle \mathbf{H} \rangle$! Treating \mathbf{E}_0 and \mathbf{H}_0 substituted into Maxwell's equations as averaged fields leads to errors which can be interpreted as *fictitious magnetization of p-lattices* and *fictitious polarization of m-lattices*. These fictitious lattice responses have no physical meaning, consequently, one should not expect the locality of TLMP.

III. EXTRACTION OF LOCAL MATERIAL PARAMETERS

In order to explain the extraction of local MP from R and T of a MTM slab, let us, first, consider an infinite lattice of electric and magnetic dipole particles. Assume, that a plane wave with wave vector $\mathbf{q} = q\mathbf{z}_0$ propagates in an infinite lattice shown in Fig. 1. The lattice is formed by orthorhombic unit cells of sizes $b \times b \times a$ (for simplicity one can skip the case of the anisotropy in the $(x-y)$ plane which gives no effect for the z -propagation).

In the left panel of Fig. 1 the particles are assumed possessing both electric and magnetic moments. In the right panel the electric and magnetic scatterers are different particles. The formulas of the local field method are the same for both these geometries. This is so since the quasistatic interaction between electric (\mathbf{p}) and magnetic (\mathbf{m}) dipoles is absent in both these structures (any p-dipole is not affected by m-dipoles lying in the same crystal plane and vice versa). Let a reference (0-numbered) p-m-particle located at the origin have electric and magnetic polarizabilities a_{ee}, a_{mm} . These polarizabilities can be resonant in the same frequency range. Let E^{loc} and H^{loc} denote the x -component of the local electric field and y -component of the local magnetic field, respectively.

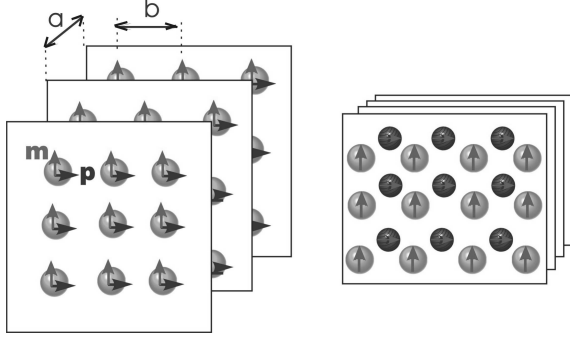


FIG. 1: Presentation of a p-m-lattice as a set of dipole crystal planes. Left: every particle has both electric and magnetic moments. Right: electric and magnetic particles are different.

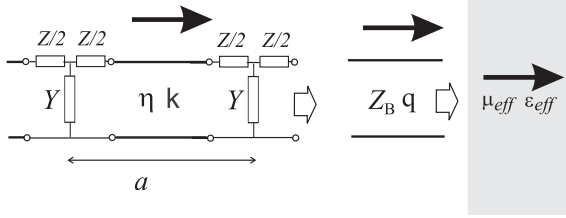


FIG. 2: Presentation of the p-m-lattice shown in Fig. 2 as (1) periodically loaded transmission line, (2) homogenized transmission line with Bloch impedance Z_B and wavenumber q (k and η are unloaded wavenumber and characteristic impedance, respectively), and (3) effective magneto-dielectric medium with non-local MP ϵ_{eff} and μ_{eff} .

Then the p-dipole and m-dipole of the particle can be expressed as:

$$p_0 = a_{ee}E^{\text{loc}}, \quad m_0 = a_{mm}H^{\text{loc}}. \quad (10)$$

Local fields are produced by other particles located at points $(x = n_x b, y = n_y b, z = n_z a)$ and can be expressed through p_0 and m_0 since $p(\mathbf{n}) = p_0 \exp(-jn_z qa)$ and $m(\mathbf{n}) = m_0 \exp(-jn_z qa)$:

$$E^{\text{loc}} = C_{ee}p_0 + C_{em}m_0, \quad H^{\text{loc}} = C_{mm}m_0 + C_{me}p_0. \quad (11)$$

Substituting 11 into 10 we come to a system

$$\frac{1}{a_{ee}}p_0 = C_{ee}p_0 + C_{em}m_0, \quad (12)$$

$$\frac{1}{a_{mm}}m_0 = C_{mm}m_0 + C_{me}p_0, \quad (13)$$

A closed-form expression for cross-coupled interaction factor $C_{em} = C_{me}$ was derived in²⁹. For $C_{ee} = C_{mm}/\eta^2$ one can find in²⁷ the expression:

$$C_{ee} = \frac{\eta\omega}{2b^2} \left(C_0 + C_{NF} + \frac{\sin ka}{\cos ka - \cos qa} \right) + j \frac{k^3}{6\pi\epsilon_m}, \quad (14)$$

where $k = \omega\sqrt{\mu_m\epsilon_m}$ is the host matrix wavenumber and $\eta = \sqrt{\mu_m/\epsilon_m}$ is its wave impedance (here ϵ_m and μ_m are absolute permittivity and permeability of the matrix). In 14 the trigonometric term describes the wave interaction of the reference p-dipole with p-dipoles of other crystal planes, the real value C_{NF} describes the near-field interaction with p-dipoles of other crystal planes. The remain of 14 is the real value C_0 which is responsible for the near-field interaction of the reference p-dipole with other p-dipoles of the reference crystal plane. For C_0 a simple expression was derived in²⁷:

$$C_0 = \frac{1}{2} \left(\frac{\cos kbs}{kbs} - \sin kbs \right),$$

where $s \approx 0.6954$. At very low frequencies $kb < 0.5$ the approximation $C_0 \approx 0.36$ is quite accurate²⁷.

Equating the determinant of 12, 13 to zero and neglecting C_{NF} (as it was shown in²⁷, it is a suitable approximation for the case $a \leq b$) we obtain the dispersion equation (an alternative equivalent form of it was derived in²⁹):

$$\cos qa = \cos ka - \sin ka \left(\frac{G}{4} + \frac{X}{4} \right) - \sqrt{\sin^2 ka \left(\frac{G}{4} - \frac{X}{4} \right)^2 + \frac{GX}{4} \sin^2 qa}. \quad (15)$$

Here the notations were introduced:

$$\left\{ \begin{matrix} G \\ X \end{matrix} \right\} = \frac{k_0 a}{\epsilon_m V \left(\left\{ \frac{1}{\frac{a_{ee}}{a_{mm}}} \right\} - j \frac{k^3}{6\pi} \left\{ \frac{1}{\frac{\epsilon_m}{\mu_m}} \right\} \right) - C_0}. \quad (16)$$

In 16 $V = ab^2$ is the unit cell volume. In the lossless structure the fundamental relations hold for electric and magnetic polarizabilities^{4,5,6}:

$$\text{Im} \frac{1}{a_{ee}} = \frac{k^3}{6\pi\epsilon_m}, \quad \text{Im} \frac{1}{a_{mm}} = \frac{k^3}{6\pi\mu_m}. \quad (17)$$

Then the parameters G and X in 15 turn out to be real-valued in lossless lattices.

Consider the averaged fields $\langle E \rangle$ and $\langle H \rangle$ defined as

$$\langle E \rangle(\mathbf{r}) = \frac{1}{V} \int_V E_x(\mathbf{r} + \mathbf{r}') d^3 \mathbf{r}', \quad (18)$$

$$\langle H \rangle(\mathbf{r}) = \frac{1}{V} \int_V H_y(\mathbf{r} + \mathbf{r}') d^3 \mathbf{r}' \quad (19)$$

Here \mathbf{r}' lies within the unit cell and E_x , H_y are components of the microscopic fields. For microscopic fields Maxwell's equations hold:

$$\text{rot} \mathbf{E} = -j\omega(\mu_m \mathbf{H} + \mathbf{M}), \quad \text{rot} \mathbf{H} = j\omega(\epsilon_m \mathbf{E} + \mathbf{P}).$$

Integrating these equations around the observation point with the use of 8 we come to Maxwell's equations for averaged fields and polarizations:

$$q < H > = \omega \varepsilon_m < E > + \omega < P >, \quad (20)$$

$$q < E > = \omega \mu_m < H > + \omega < M >, \quad (21)$$

whose solutions take form

$$\begin{cases} < E >, < H > \\ < P >, < M > \end{cases} = \begin{cases} E_L, & H_L \\ P_L, & M_L \end{cases} \exp(-jqz). \quad (22)$$

The amplitudes E_L , H_L are related with Bloch harmonics E_n , H_n of transversely averaged fields as

$$\begin{cases} E_L \\ H_L \end{cases} = \begin{cases} E_0/qa \\ H_0/qa \end{cases} + \sum_{n=1}^{\infty} \begin{cases} E_n/(qa + 2\pi n) \\ H_n/(qa + 2\pi n) \end{cases} \quad (23)$$

and at the reference particle center $x = y = z = 0$ we have $< P > = p_0/V$ and $< M > = m_0/V$. Equations 20 and 21 at $z = 0$ are equivalent to relations

$$q^2 = \omega^2 \varepsilon_L \mu_L, \quad Z_L \equiv \frac{< E > (0)}{< H > (0)} = \sqrt{\frac{\mu_L}{\varepsilon_L}}. \quad (24)$$

Here coefficients ε_L and μ_L are defined by formulas

$$\begin{cases} < P > (0) \\ < M > (0) \end{cases} = \begin{cases} (\varepsilon_L - \varepsilon_0) < E > (0) \\ (\mu_L - \mu_0) < H > (0) \end{cases} \quad (25)$$

and have physical meaning of absolute permittivity and permeability. They should be *local* even at *moderately* low frequencies 1 since relate *averaged* fields and *averaged* polarizations instead of zero-order Bloch harmonics. It is difficult to prove strictly this fact (see also the discussion in³⁰), but the explicit examples confirm it.

Let us assume that we know parameters G and X describing the crystal plane response. The evident way to find local MP is to express polarizabilities a_{ee} , a_{mm} through G , X using formulas 16 and then to apply the well-known Clausius-Mossotti formula expressing ε_L via a_{ee} and μ_L via a_{mm} . However, this formula is quasi-static and is hardly accurate at moderate frequencies. Formulas 24 show another (not static) way for obtaining local MP through G and X . Really, from 12, 13 we have

$$\eta p_0 \left(\frac{2}{G} - \frac{\sin ka}{\cos qa - \cos ka} \right) = m_0 \frac{\sin qa}{\cos qa - \cos ka},$$

$$m_0 \left(\frac{2}{X} - \frac{\sin ka}{\cos qa - \cos ka} \right) = \eta p_0 \frac{\sin qa}{\cos qa - \cos ka}.$$

These relations give the auxiliary coefficient one needs in order to find Z_L :

$$\gamma = \frac{\eta p_0}{m_0} = \frac{\eta < P > (0)}{< M > (0)} =$$

$$\sqrt{\frac{G}{X}} \sqrt{\frac{\cos qa - \cos ka - \frac{G \sin ka}{2}}{\cos qa - \cos ka - \frac{X \sin ka}{2}}}. \quad (26)$$

From 25 we obtain (see also in^{29,30}:

$$\gamma = \eta Z_L \frac{(\varepsilon_L - \varepsilon_0)}{(\mu_L - \mu_0)}.$$

However, from 24 one has

$$\varepsilon_L = \frac{q}{\omega Z_L}, \quad \mu_L = \frac{q}{\omega} Z_L. \quad (27)$$

Therefore the wave impedance Z_L can be expressed as

$$\frac{Z_L}{\eta} = \frac{\gamma k + q}{\gamma q + k}. \quad (28)$$

So, from G , X one can find q , then γ , then Z_L , and finally local MP.

Practically, what is suggested is the generalization of the quasistatic approach based on the Clausius-Mossotti formula to the range of moderate frequencies. In the quasistatic limit our result for ε_L and μ_L must be the same as this classical result. Let us prove it for simplicity for a cubic p-lattice. In this case $X = 0$ and 15 simplifies to

$$\cos(qa) = \cos(ka) - \frac{G}{2} \sin(ka). \quad (29)$$

Substitution of $\gamma \rightarrow \infty$ into 28 gives $Z_L = \eta k/q$ and $q = k_0 \sqrt{\varepsilon_L}$. For low frequencies ($ka \ll \pi$) we take in account two terms of the Taylor expansion of trigonometric functions in 29 and substituting expression 16 for G obtain:

$$1 - \frac{(k_0 a)^2 \varepsilon_L}{2\varepsilon_0} = 1 - \frac{(k_0 a)^2 \varepsilon_m}{2\varepsilon_0} - \frac{(k_0 a)^2 \varepsilon_m \left(1 - \frac{(k_0 a)^2 \varepsilon_m}{6\varepsilon_0} \right)}{2 \left[V \varepsilon_m \operatorname{Re} \left(\frac{1}{a_{ee}} \right) - C_0 \right]}.$$

Neglecting the small term $(ka)^2/6 \ll 1$ we obtain the Clausius-Mossotti equation for local permittivity:

$$\varepsilon_L = \varepsilon_m + \frac{1}{V \operatorname{Re} \left(\frac{1}{a_{ee}} \right) - \frac{C_0}{\varepsilon_m}}.$$

Our factor $C_0 = 0.36$ differs slightly from classical 0.33. This difference is related with the ignorance of near-field interaction between adjacent crystal planes. The needed correction can be easily introduced in the theory. Taking in account three terms in the Taylor expansion of cosine and sine functions gives the frequency corrections of the order $(k_0 a)^2$ and $(k_0 a)^3$ to the Clausius-Mossotti relation. These corrections were obtained in another way in work²⁴. A similar proof can be done for $p-m$ -lattices where one comes in the limit case $(ka, qa) \ll \pi$ to the Clausius-Mossotti formulas for both ε_L and μ_L .

Now, let us relate ε_L , μ_L with R and T coefficients of a finite slab. The simplest way is to find q and $Z = Z_B$ entering 3 inverting 2 in a usual way and then find G and X through them. With G , X we can calculate local MP. Let us then relate Bloch impedance Z_B with G and X . For it one can apply the transfer matrix method. For the case under consideration (the normal propagation with respect to $(x-y)$ crystal planes) the transfer matrix of a lattice unit cell \mathbf{F} with components (A, B, C, D) is defined as

$$\begin{bmatrix} E_{TA}(-\frac{a}{2})/\eta \\ H_{TA}(\frac{a}{2}) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_{TA}(\frac{a}{2})/\eta \\ H_{TA}(\frac{a}{2}) \end{bmatrix}. \quad (30)$$

It can be calculated as a product of transfer matrices of its three portions: two $a/2$ -long pieces of the TL modeling the host medium and the lumped circuit modeling the p-m-crystal plane (see Fig. 2):

$$\mathbf{F} = \mathbf{F}_{TL}\mathbf{F}_{YZ}\mathbf{F}_{TL}, \quad (31)$$

Here

$$\mathbf{F}_{TL} = \begin{bmatrix} \cos(\frac{ka}{2}) & j \sin(\frac{ka}{2}) \\ j \sin(\frac{ka}{2}) & -\cos(\frac{ka}{2}) \end{bmatrix}, \quad \mathbf{F}_{YZ} = \begin{bmatrix} 1 & Z \\ Y & -1 \end{bmatrix}. \quad (32)$$

Values $G = -jY$ and $X = -jZ$ are parameters we introduced above as G and X and are given by 16. For a lossless structure G and X are equal to the shunt admittance and series reactance of the loading, respectively. Parameters Y and Z are known in the theory of planar grids as the grid electric admittance and the grid magnetic impedance and describe the ratio between tangential electric and magnetic fields in the grid plane and electric and magnetic surface currents induced in the grid (all averaged over the grid periods). Even if particles have sizes comparable to the period a one can attribute effective currents to the crystal plane $z = 0$. This procedure is basically the same as replacing the finite particles by point p- and m-dipoles which is an excellent approximation for many practical cases.

The Bloch impedance is given by relation²⁸:

$$Z_B = \frac{B}{\sin qa}. \quad (33)$$

After some algebra we obtain from 31-33:

$$Z_B = \frac{\sin qa}{G \cos^2(ka/2) - X \sin^2(ka/2) + \sin ka}. \quad (34)$$

If one knows q and Z one can express G and X through them using equations 15 and 34. Then one finds local MP through G and X using 27, 28 and 26.

To conclude this section one can notice that studying the transfer matrix of a cascade of lattice unit cells it is possible to prove that TLMP^{1,19,21} are not mesoscopic. Being found from R and T of the layer containing N lattice unit cells across it these MP can be applied for slabs with arbitrary thickness (if divisible by a). Therefore,

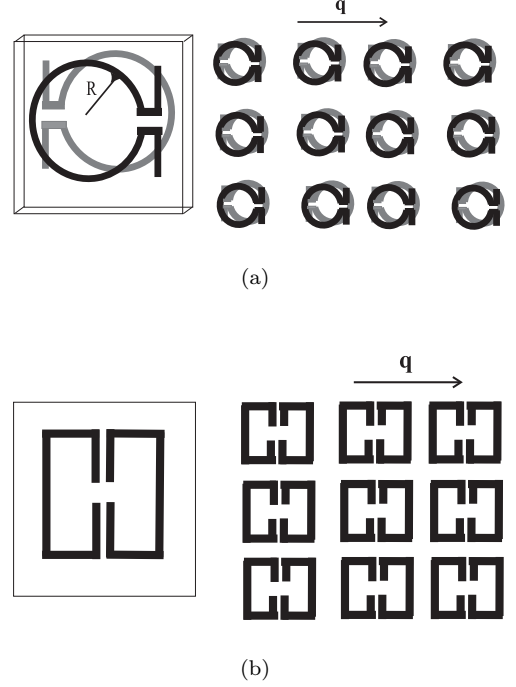


FIG. 3: The lattice of pairs of Ω -particles (a), and the lattice of infrared SRRs suggested by J.B. Pendry and S. O'Brien in⁸ (b).

one can attribute TLMP found for finite slabs to infinite lattices. Considering the unit cell transfer matrix \mathbf{F} one can also prove that constitutive parameters suggested in works^{17,18} and discussed in¹⁶ are equivalent to TLMP.

IV. RESULTS

To illustrate the locality of one set of MP and non-locality of the other one, two numerical examples of the direct calculation of them through the known polarizabilities of particles are presented. The algorithm of this calculation is as follows:

$$(a_{ee}, a_{mm}) \rightarrow (G, X) \rightarrow \begin{cases} (q, Z_B) \rightarrow (\varepsilon_{\text{eff}}, \mu_{\text{eff}}) \\ (q, Z_L) \rightarrow (\varepsilon_L, \mu_L) \end{cases}. \quad (35)$$

The first arrow implies formulas 16, the second one implies the solution of 15 and the use of formula 33 in the upper case and 34 in the lower case. To find local MP formulas 26–28 were used. The first numerical example corresponds to a cubic lattice formed by capacitively loaded wire dipoles. The second example corresponds to a lattice of pairs of Ω -particles as is shown in Fig. 3 (a). In the first example the polarizability of particles is nearly static $a_{ee} \approx l^2 C_0$, where l is the effective length of a wire dipole and C_0 is the loading capacitance. However, with rather large C_0 the effect of the presence of non-resonant

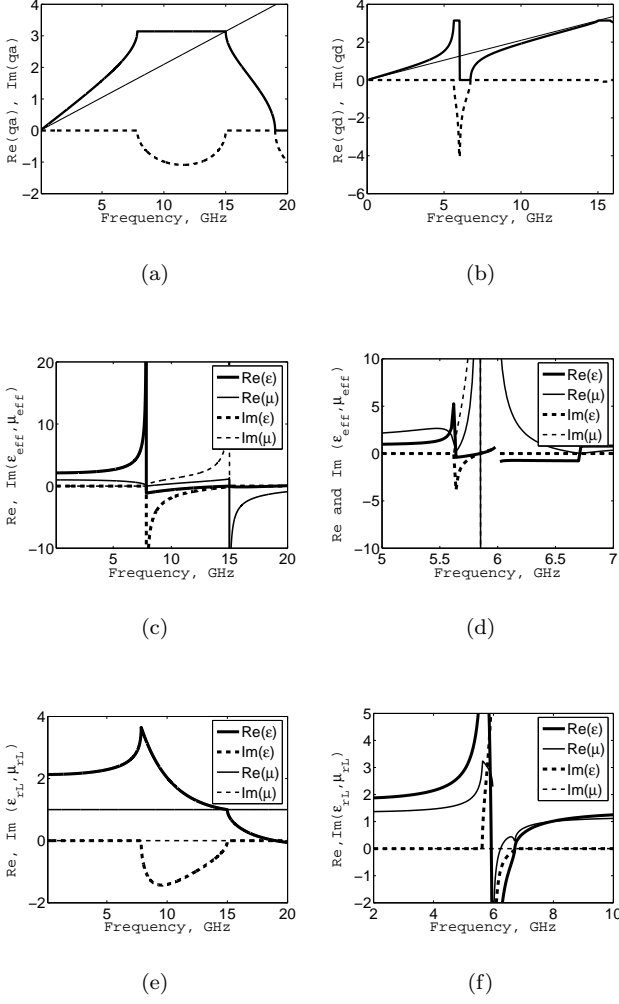


FIG. 4: (a)–dispersion in a lattice of non-resonant dipoles; (b)–dispersion in a lattice of resonant p-m-particles (pairs of Ω -particles); (c)–non-local MP for the lattice of non-resonant dipoles; (d)–non-local MP for p-m-particles; (e)–local MP for the lattice of non-resonant dipoles; (f)–local MP for the lattice of resonant p-m-particles.

dipoles can be significant. This effect is a wide stop-band with staggered mode near the first lattice resonance as one can see in Fig. 4 (a). Notice, that qualitatively same results as shown in Fig. 4 (a,c,e) should correspond also to lattices of metal spheres of radius 2 – 4 mm which also behave below 15 GHz (the lattice period was chosen $a = 10$ mm in both examples and the host matrix was free space) as non-resonant dipoles with rather high static polarizability. A lattice of electric dipoles cannot have local magnetic susceptibility. And the relative local permeability μ_{rL} is identically unity in Fig. 4 (e). The locality conditions are satisfied until the 1st lattice resonance which occurs at 7.5 GHz. Non-local MP in Fig. 4 (c) contain non-trivial permeability and imaginary parts

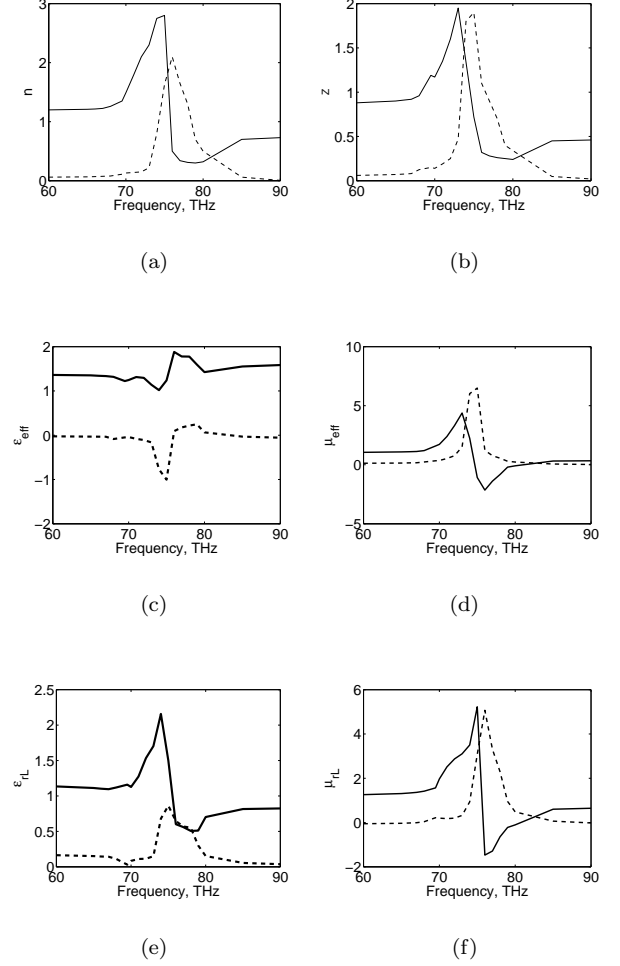


FIG. 5: (a)–refraction index extracted in¹⁰ from R , T coefficients of a slab comprising the lattice of silver SRRs with period $a = 600$ nm; (b)–normalized Bloch impedance extracted in¹⁰; (c)–non-local permittivity extracted for this lattice; (d)–non-local permeability; (e)–local permittivity; (f)–local permeability.

of ε_{eff} and μ_{eff} have opposite signs. Even below 7.5 GHz we observe in Fig. 4 (c) the violation of causality.

In the second example both electric and magnetic polarizabilities of doubled Ω -particles are resonant. We calculate them using formulas^{29,31}:

$$a_{ee} = \frac{A}{\omega_0^2 - \omega^2 + j\omega\Gamma}, \quad A = \frac{l^2}{L_0}, \quad (36)$$

$$a_{mm} = \frac{B\omega^2}{\omega_0^2 - \omega^2 + j\omega\Gamma}, \quad B = \frac{\pi^2 \mu_0^2 R^4}{L_0}, \quad (37)$$

where L_0 is the inductance of metal rings, l is the effective length of the electric dipole induced in the particle, R is the ring radius shown in Fig. 3. In this example the

p-m-particles resonate at 6 GHz. Loss factor Γ was taken negligibly small in order to avoid the mess in complex solutions of equation 15. The second example is illustrated by plots in Fig. 4 (b,d,f). The staggered mode holds in the narrow band 5.8...6 GHz. In this band local MP have no physical meaning. Only within this band the locality of ε_{rL} , μ_{rL} is lost. Outside it (even at higher frequencies) the sign of $\text{Im}(\varepsilon_{rL}, \mu_{rL})$ is correct and the frequency behavior of $\text{Re}(\varepsilon_{rL}, \mu_{rL})$ in the low-loss region is causal.

In our third example we study the structure for which the input data are R and T coefficients of a slab. It is filled by a lattice of silver split-ring resonators (SRRs) shown in Fig. 3 (b). This structure operating in the infrared range was studied in^{8,10}. For this case the algorithm is as follows:

$$(R, T) \rightarrow (q, Z_B) \rightarrow (G, X) \rightarrow (\varepsilon_L, \mu_L). \quad (38)$$

In algorithm 38 the first arrow corresponds to the inversion of formulas 2 and 3. However, q and Z_B were already found in^{8,10} and one can use the data presented in P. 56 of¹⁰ for $n = q/k_0$ and Z_B . In the second step of 38 one can use formulas 15 and 34 solving them as an equation for G and X and finally find local MP from 26–28.

To calculate the MP in the third example the geometry of infrared SRRs are not important since the input data are values of R and T . The period a in this case is equal $a = 600$ nm and the time dependence is $\exp(-i\omega t)$, as it is adopted in optics. The input data taken from¹⁰

are shown in Fig. 5 (a,b) and the results are presented in Fig. 5 (c-f). The difference between the results obtained for local and non-local material parameters confirms the theory developed in the present paper. Parameters extracted by a direct inversion of Fresnel formulas 2 and 3 are non-local and parameters obtained using the suggested theory satisfy to the locality conditions.

V. CONCLUSION

In this paper it is demonstrated that local material parameters of lattices can be introduced not only at very low frequencies but also in the region of moderately low frequencies where MTM operate. These local MP complement a pair of non-local MP which were obtained before in known works. Though at moderate frequencies local MP allow to solve boundary problems for MTM slabs only at expense of introducing two transition layers at the sides of the slab, these MP are very important. Only these MP do not depend on the incidence angle and only they can be used to study the interaction of MTM with wave packages and evanescent waves. It is then important to learn to extract these parameters from R and T coefficients of MTM slabs. The simplest algorithm of this extraction is explained in the present paper, and the numerical examples show that the obtained MP satisfy to locality conditions.

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- ¹ C. Caloz and T. Itoh, *Electromagnetic metamaterials: transmission line theory and microwave applications* (NY, J. Wiley and Sons, 2006)
 - ² S. Anantha Ramakrishna, Rep. Prog. Phys. **68**, 449521 (2005)
 - ³ L.D. Landau and E.M. Lifshitz, *Electrodynamics of continuous media* (Oxford, Pergamon Press, 1960)
 - ⁴ M. Planck, Sitzungsber. Konig. Preuss Acad. **24**, 470 (1902); see also ibid. **24**, 480 (1904)
 - ⁵ L. Mandelstam, Phys. Z. **9**, 641 (1908)
 - ⁶ J. Sipe and J. V. Kravendok, Phys. Rev. **A 9**, 1806 (1974)
 - ⁷ D. R. Smith, D. Shurig, Phys. Rev. Lett. **90**, 077405 (2003)
 - ⁸ S. O'Brien and J. B. Pendry, J. Phys. Condens. Matter, **14**, 6383 (2002)
 - ⁹ S. O'Brien, D. MacPeake, S. A. Ramakrishna and J. B. Pendry, Phys. Rev. B, **69**, 241101 (2004)
 - ¹⁰ S. O'Brien, *Artificial Magnetic Structures*, PhD Thesis, University of London and Imperial College, 2002. Available at www.imperial.ac.uk/research/cmth/research/theses/S.O'Brien.pdf
 - ¹¹ D. R. Smith, P. Kolinko, and D. Schurig, J. Opt. Soc. Am. B **21**, 1032 (2004)
 - ¹² K. C. Huang, M. L. Povinelli, and J. D. Joannopoulos, Applied Physics Lett. **85**, 543 (2004)
 - ¹³ N. Katsarakis, T. Koschny, M. Kafesaki, E. N. Economou, C. M. Soukoulis, Appl. Phys. Lett. **84**, 2943 (2004)
 - ¹⁴ N. Katsarakis, G. Konstantinidis, A. Kostopoulos, et al. Optics Letters, **30**, 1348 (2005)
 - ¹⁵ M. Born and Kun Huang, *Dynamic theory of crystal lattices* (Oxford, Oxford Press, 1954)
 - ¹⁶ D. Smith and J. B. Pendry, JOSA B, **23**, 391 (2006)
 - ¹⁷ J. B. Pendry, A. J. Holden, D. J. Robins, and W. J. Stewart, J. Phys. Condens. Matter, **10**, 4785 (1998)
 - ¹⁸ J. B. Pendry, A. J. Holden, D. J. Robins, and W. J. Stewart, IEEE Trans. MTT, **47** 2075 (1999)
 - ¹⁹ A. Grbic and G. V. Eleftheriades, Phys. Rev. Lett., **92** 117403 (2004)
 - ²⁰ G. V. Eleftheriades, A. K. Iyer and P. C. Kremer, IEEE Trans. Microwave Theory Tech., **50** 2702 (2002)
 - ²¹ C. Caloz and T. Itoh, IEEE Microwave Wireless Compon. Lett., **13**, 547 (2003)
 - ²² C. R. Simovski, S. He, M. Popov, Phys. Rev. B, **62**, 13718 (2000)
 - ²³ C. R. Simovski, S. A. Tretyakov, A. H. Sihvola and M. Popov Eur. Phys. Journal: Applied Physics, **9**, 233 (2000)
 - ²⁴ C. R. Simovski, Weak spatial dispersion in composite media, (St. Petersburg, Politekhnik Edition, 2003, in Russian)
 - ²⁵ C. R. Simovski and B. Sauviac, Eur. Phys. J.: Appl. Phys. **17**, 11 (2002)
 - ²⁶ P.A. Belov and C.R. Simovski Phys. Rev. B, **73**, 045102 (2006)
 - ²⁷ S. A. Tretyakov, *Analytical Modeling in Applied Electromagnetic* (Norwood, MA, Artech House, 2003)
 - ²⁸ D. M. Pozar, *Microwave Engineering* (NY, John Wiley

and Sons, 1998)

- ²⁹ C. Simovski and S. He, Phys. Lett. **A 311**, 254 (2003)
- ³⁰ C. Simovski, P.A.Belov, and S. He, IEEE Trans. Antennas Propagat. **51**, 2582 (2003)
- ³¹ A. Serdyukov, I. Semchenko, S.A. Tretyakov, A. Sihvola,

Electromagnetics of Bi-anisotropic Materials, Theory and Applications (NY, Gordon and Breech Science Publishers, 2001)

